J. Fluid Mech. (1981), *vol.* 113, *pp.* 469–485
Printed in Great Britain

Planform selection by finite-amplitude thermal convection between poorly conducting slabs

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(Received 26 January 1981 and in revised form 8 July 1981)

Three-dimensional finite-amplitude thermal convection in a fluid layer is considered in the case where the boundaries of the layer are much poorer conductors than the fluid. It can be shown that if the conductive heat flux through the layer is not too large, the horizontal scale of motion is much greater than the layer depth. Then a 'shallow water theory' approximation leads to a nonlinear evolution equation for the leading-order temperature perturbation, which can be analysed in terms of a variational principle. It is proved that the preferred planform of convection is a square cell tesselation, as found in a rather more restricted parameter range by Busse & Riahi (1980), in contrast to the roll solutions that obtain for perfectly conducting boundaries. It is also shown that the preferred wavelength of convection increases slowly with amplitude.

1. Introduction

The mechanisms by which a convectively unstable fluid layer chooses the horizontal structure of the velocity field that transmits the heat across the system have been extensively studied, both theoretically and experimentally, in recent years. Most attention has been focused on the behaviour of the convection when the amplitudes are not very large and the flow is relatively ordered. Then not only is discussion of a 'pattern' meaningful, but the theoretical problem is often amenable to perturbation theory. Most analysis has been undertaken in the idealized situation for which the boundaries of the fluid layer are taken to be excellent conductors of heat, on which the temperature is almost uniform (Schlüter, Lortz & Busse 1965; Palm 1960; Clever & Busse 1974; Newell & Whitehead 1969; for further references see the recent review by Busse 1978). The principal result of the calculations is that for an effectively infinite layer the stable form of convection is rolls (in which the velocity field is everywhere orthogonal to a given direction), except for a very small region around marginal stability where effects of temperature dependent viscosity etc. lead to subcritical convection with a hexagonal planform.

In many laboratory experiments, however, the working fluid is surrounded by a layer of material (often Plexiglas) which is no better a conductor of heat than the fluid and is often (especially for low-Prandtl-number fluids like mercury) rather worse. Then the condition of uniform temperature on the boundaries must be relaxed, and this can dramatically affect, not only the preferred horizontal length scale of convection, but also its planform. These considerations are not purely academic; they may be relevant, for example, to understanding the form of convection in the Earth's upper mantle (see, for example, Chapman, Childress & Proctor 1980). It is natural, then, to investigate the other 'extreme' problem, that of almost insulating boundaries. Linearized stability theory for convection between layers that are much poorer conductors than the fluid (Sparrow, Goldstein & Jonsson 1964; Hurle, Jakeman & Pike 1967) shows that the 'most unstable' wavelength (the one requiring the lowest temperature difference to destabilize it) becomes very large as the 'Biot number', the dimensionless parameter that measures the relative conductivity of boundary to fluid, tends to zero. Chapman (1978, 1980) and Chapman & Proctor (1980, hereinafter referred to as CP) considered nonlinear convection at zero Biot number using a 'shallow water theory' technique first used for convection problems by Childress & Spiegel (1981). CP considered two-dimensional (roll) solutions only, but were able to prove rigorously that for zero Biot number the convection selected the largest wavelength available to it, even though the wavelength of maximum growth rate on linear theory was much shorter. Busse & Riahi (1980) considered weakly nonlinear three-dimensional solutions at small Biot number, utilizing the fact that the most unstable wavelength is very long in this case. By means of analysis similar to that of Schlüter *et al.* (1965) they showed that *square* convection cells are the stable planform. Poyet (1980) has discussed the small-Biot-number problem by using the Childress & Spiegel (1981) method. He concentrates on two-dimensional solutions with special boundary conditions, and finds interesting time-dependent behaviour. This behaviour does not occur in our geometry. Depassier & Spiegel (1981*a*, *b*) have discussed the limitations on the Boussinesq approximation in the fixed-heat-flux problem, and Poyet & Spiegel (1982) have discussed aspects of the transition from small to large Biot number in the non-Boussinesq case.

In the present paper we apply the expansion method developed in CP to the three-dimensional convection problem at small Biot number, in the special case when the depth of the conducting slabs is of the same order as the depth of the fluid layer. The main thrust of the analysis is to determine the planform of convection that is actually realized, among the many permitted by linearized theory. The method allows the consideration of much larger amplitudes than those treated by Busse and Riahi. The nonlinear stability problem can be reduced to the determination of the minima of a certain functional that is a generalization of the one used in CP (and was originally proposed by S. Childress). It can then be shown quickly and rigorously that for small Biot number roll solutions are unstable over a wide range of amplitudes, and (after some computation) that square-cell solutions are indeed the stable ones over the entire range of validity of the approximation. It can also be demonstrated that the 'preferred' wavelength of convection increases slowly with amplitude. Convective amplitudes in the Busse & Riahi (1980) study were too small to permit any statement about the change in wavenumber from its 'most unstable' value.

The structure of the paper is as follows. In § 2 the problem is formulated. Linearized stability theory is considered in § 3, where it is shown that the most unstable wavenumber tends to zero as the Biot number tends to zero. Section 4 develops the scaling for the nonlinear problem, and the linear theory of the reduced system is considered in § 5. The nonlinear variational principle is introduced in § 6, the main results are described in § 7 and conclusions are presented in § 8.

2. Formulation

We consider a layer of Boussinesq fluid of depth $2d$ lying between solid poorly conducting slabs of depth λd, where λ is of order unity. The conductivity κ_1 of the slabs is much less than κ, the conductivity of the fluid. Cartesian co-ordinates are chosen with origin at the mid point of the layer. Gravity $\mathbf{g} = -g\hat{\mathbf{z}}$ is perpendicular to the boundaries; the fluid has velocity $\mathbf{u}$, pressure p and kinematic viscosity ν, where it is supposed that $\nu \gg \kappa$, so that the Prandtl number $\sigma = \nu/\kappa$ is effectively infinite and the effects of fluid inertia can be neglected. It will emerge later, though, that our analysis is independent of σ provided it is not too small; see § 8. The temperature T at the top and bottom of the 'sandwich' is fixed so that in the absence of motion the temperature gradient in the fluid $\partial T/\partial z = -q$. Thus when $\mathbf{u} = 0$,

$$T = T_0 - qz, \quad |z| \leqslant d,$$
$$T = T_0 \mp qd(1-\zeta^{-1}) - q\zeta^{-1}z, \begin{Bmatrix} d \leqslant z \leqslant d(1+\lambda) \\ -d \geqslant z \geqslant -d(1+\lambda) \end{Bmatrix}, \text{ respectively.} \tag{2.1}$$

where $\zeta = \kappa_1/\kappa$ and T_0 is a reference temperature at which the density is ρ_0. If the temperature perturbation due to any fluid motion is $\theta(\mathbf{x}, t)$ in the fluid and $\tilde{\theta}(\mathbf{x}, t)$ in the solid, and α the coefficient of thermal expansion, the non-dimensional equations of motion and heat conduction are:

$$\nabla p = R\theta\hat{\mathbf{z}} + \nabla^2\mathbf{u}, \quad \frac{\partial\theta}{\partial t} + \mathbf{u}\,.\,\nabla\theta = \mathbf{u}\,.\,\hat{\mathbf{z}} + \nabla^2\theta, \quad \nabla\,.\,\mathbf{u} = 0 \tag{2.2}$$

in the fluid; and

$$\frac{\partial\tilde{\theta}}{\partial t} = \zeta\nabla^2\tilde{\theta}, \tag{2.3}$$

in the slabs. In equations (2.2), (2.3) $|\mathbf{u}|$ is scaled with κ/d, p with $\rho_0\nu\kappa/d^2$, time t with d^2/κ, lengths with d, and $\theta, \tilde{\theta}$ with qd. The dimensionless parameter R (the Rayleigh number) is defined by

$$R = g\alpha q d^4/\kappa\nu. \tag{2.4}$$

The velocity vanishes on the boundaries $z = \pm 1$, and the boundary conditions on the temperature are

$$\tilde{\theta} = 0, \quad z = \pm(1+\lambda), \tag{2.5}$$

$$\theta = \tilde{\theta}, \quad z = \pm 1, \tag{2.6}$$

$$D\theta = \zeta D\tilde{\theta}, \quad z = \pm 1, \tag{2.7}$$

where $D\theta = \partial\theta/\partial z$ etc. As previously noted, we shall suppose that ζ is small, that $\lambda \gg \zeta$ and that λ is not so large that the depth of the slabs is comparable with the horizontal scale of convection. The object of the analysis to follow is to examine the nature of the instability that arises if R is sufficiently large. We shall make no assumptions concerning the evolved velocity and temperature fields except to require that they be periodic in x and y, thus giving the tesselated cellular structure observed often in experiments. It is well known that in the absence of inertial effects there is no source of vertical vorticity in the fluid, which therefore decays to zero. We may thus represent the velocity field $\mathbf{u}$ in terms of the single scalar $\phi(\mathbf{x}, t)$ by

$$\mathbf{u} = \nabla \wedge \nabla \wedge (\phi\hat{\mathbf{z}}). \tag{2.8}$$

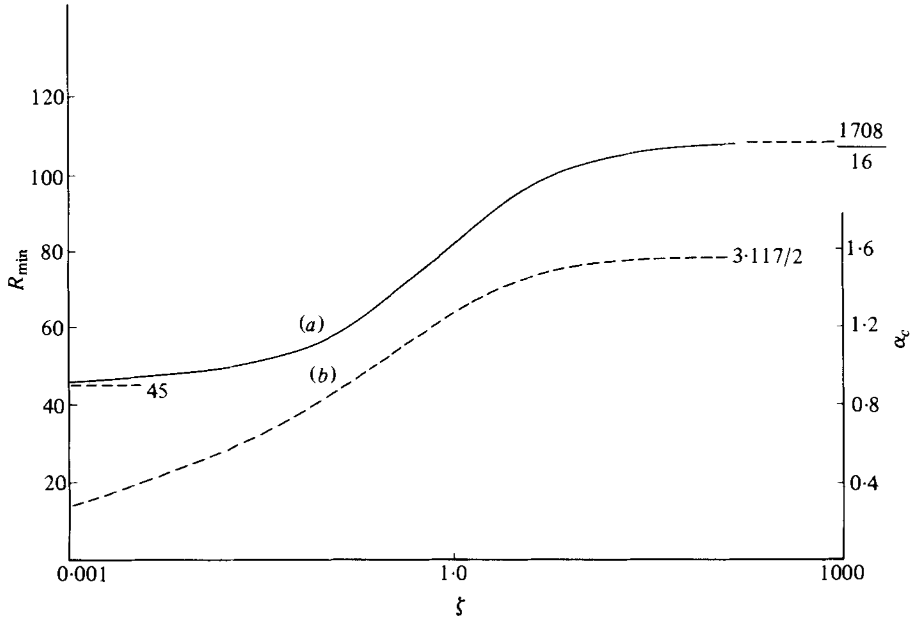

FIGURE 1. Graphs of (a) R_{min} and (b) α_c as functions of ζ for $\lambda = 1$.

Thus $\mathbf{u}$ is solenoidal and $\hat{\mathbf{z}} \,.\, \nabla \wedge \mathbf{u} = 0$, as required. If this representation is substituted into the governing equations (2.2) they may be rewritten as

$$0 = R\theta - \nabla^4\phi, \tag{2.9}$$

$$\frac{\partial\theta}{\partial t} + \nabla_H(D\phi) \,.\, \nabla_H\theta - \nabla_H^2\phi D\theta = -\nabla_H^2\phi + \nabla^2\theta, \tag{2.10}$$

where $\nabla_H = (\partial/\partial x, \partial/\partial y, 0)$. The boundary conditions on ϕ at the top and bottom of the layer are

$$\phi = D\phi = 0, \quad z = \pm 1. \tag{2.11}$$

3. Linearized stability theory

If we ignore the nonlinear terms in (2.10) we are left with a linear system separable in x, y and t, that describes the evolution of small disturbances. It may easily be shown that if $\mathbf{u}, \theta, \tilde{\theta} \propto e^{st}$ then s must be real, and so the boundary between growing and decaying solutions is given by $s = 0$. If we write $\phi = \hat{\phi}(z) f(x,y)$, $\theta = \hat{\theta}(z) f(x,y)$, $\tilde{\theta} = \hat{\tilde{\theta}}(z) f(x,y)$ where $\nabla_H^2 f = -\alpha^2 f$, the equations for steady fields become

$$\left.\begin{aligned} (D^2-\alpha^2)\hat{\tilde{\theta}} &= 0, \\ 0 &= R\hat{\theta} - (D^2-\alpha^2)^2\hat{\phi}. \\ 0 &= \alpha^2\hat{\phi} + (D^2-\alpha^2)\hat{\theta}. \end{aligned}\right\} \tag{3.1}$$

Thus the critical value $R = R_0$ for linear instability is a function of ζ, λ and α^2, determined as an eigenvalue of (3.1). Of particular interest is the behaviour of R_0 as a function of α^2 when ζ and λ are fixed, and in particular the minimum $R_{min}(\zeta, \lambda)$ that occurs at $\alpha = \alpha_c(\zeta, \lambda)$. Sparrow *et al.* (1964) studied the present problem in the limit $\lambda \to 0$, $\zeta\lambda \to$ constant, equivalent to a 'Newton's Law of Cooling' boundary condition; here we give results for $\lambda = 1$ (which is typical) and plot $R_{\min}$ and α_c as a function of ζ (for a summary of the method, see the appendix). As expected, when $\zeta \to \infty$, $R_{\min} \to 1708/16$ and $\alpha_c \to 3{\cdot}117/2$, in agreement with the classical result. [The factors 16 and 2 are due to our layer being of depth $2d$.] For small ζ, we find that $R_{\min} = 45 + O(\zeta^{\frac{1}{2}})$, $\alpha_c = O(\zeta^{\frac{1}{4}})$ in agreement with the results for the singular case $\zeta = 0$ given by CP. Furthermore, it is clear from the extremal property of $R_{\min}$ that when $R - R_{\min} \sim \zeta^{\frac{1}{2}}$, the range of wavenumbers above the (R_0, α) curve has width $O(\zeta^{\frac{1}{4}})$. [See figure 1].

4. Scaling and derivation of the canonical equation

The theory given above for general ζ suggests that the behaviour of the nonlinear solutions when ζ is small can be obtained by exploiting the fact that all growing disturbances have long horizontal scales compared with the layer depth. Thus horizontal derivatives are small, and we write

$$(\partial/\partial x, \partial/\partial y) = \epsilon(\partial/\partial X, \partial/\partial Y), \quad \epsilon \ll 1. \tag{4.1}$$

Guided by the results of the previous section, we then scale other relevant quantities by writing

$$R - R_c = \mu^2\epsilon^2, \quad \zeta = \hat{\zeta}\epsilon^4, \quad t = \tau\epsilon^{-4} \quad [\mu, \hat{\zeta} = O(1)] \tag{4.2}$$

where we write $R_c = 45$ for convenience.† The scalings for x, y, t and R are just those used by CP for the case $\zeta = 0$. No assumptions are made at this stage about the magnitudes of θ, $\tilde{\theta}$ and ϕ, but we formally expand them in powers of ϵ^2;

$$\left.\begin{aligned} \theta &= \theta_0(X, Y, z, \tau) + \epsilon^2\theta_2(X, Y, z, \tau) + \ldots, \\ \tilde{\theta} &= \tilde{\theta}_0(X, Y, z, \tau) + \epsilon^2\tilde{\theta}_2(X, Y, z, \tau) + \ldots, \\ \phi &= \phi_0(X, Y, z, \tau) + \epsilon^2\phi_2(X, Y, z, \tau) + \ldots. \end{aligned}\right\} \tag{4.3}$$

We then substitute (4.1)–(4.3) into (2.3), (2.9) and (2.10) and attempt to solve the sequence of problems that emerge. The method is fully explained in CP, where it is noted that it is completely analogous to 'shallow-water theory'; the first use of the technique for problems in convection theory was made by Childress & Spiegel (1981) in their study of the instability due to negatively geotactic swimming micro-organisms. Only a summary of the development is given here.

First we treat the problem in the slabs (equation (2.3)); only the leading-order equation is needed here, and it is

$$\frac{\partial\tilde{\theta}_0}{\partial\tau} = \hat{\zeta}D^2\tilde{\theta}_0. \tag{4.4}$$

At $z = \pm(1+\lambda)$, $\tilde{\theta} = 0$, while at $z = \pm 1$, $D\theta = \epsilon^4\hat{\zeta}D\theta_0$, so that

$$D\theta_0 = D\theta_2 = 0; \quad D\theta_4 = \hat{\zeta}D\tilde{\theta}_0, \quad z = \pm 1. \tag{4.5}$$

† It is clear that (4.2) does not completely define ϵ. We retain the representation given since it is sometimes convenient to consider the effect of changing R at fixed ζ, and sometimes more illuminating to do the opposite.

For the fluid, at leading order, the temperature equation (2.10) yields

$$D^2\theta_0 = 0 \tag{4.6}$$

with the boundary condition (4.5); (4.6) is solved by

$$\theta_0 = f(X, Y, \tau). \tag{4.7}$$

We then continue the problem solving sequence until we find an equation that determines f. Substitution of (4.7) into (2.9) at $O(1)$ gives

$$D^4\phi_0 = R_c f, \quad \phi_0 = D\phi_0 = 0, \quad z = \pm 1, \tag{4.8}$$

and this has the solution

$$\phi_0 = -R_c P(z) f, \tag{4.9}$$

where $P(z) = -\frac{1}{24}(1-z^2)^2$. [The notation is chosen to be consistent with that of CP.] Then (2.10) at $O(\epsilon^2)$ becomes

$$-R_c DP|\nabla_H f|^2 = (1+R_c P)\nabla_H^2 f + D^2\theta_2, \tag{4.10}$$

where now $\nabla_H = (\partial/\partial X, \partial/\partial Y)$. This inhomogeneous boundary-value problem has a solution if and only if

$$\int_{-1}^{1} D^2\theta_2\, dz = 0.$$

Thus we must have

$$\int_{-1}^{1} (1+R_c P)\, dz = 0; \quad \text{hence } R_c = 45\text{, as required.} \tag{4.11}$$

Equation (4.10) can then be solved for θ_2 and the result substituted into (2.9) at $O(\epsilon^2)$ to obtain ϕ_2; on returning to (2.1) at $O(\epsilon^4)$ we obtain another inhomogeneous equation for θ_4 whose solvability condition is

$$\int_{-1}^{1} D^2\theta_4\, dz = 2\xi D\tilde{\theta}|_{z=1}, \tag{4.12}$$

where the symmetry of $\tilde{\theta}$ about $z = 0$ has been taken into account. When the appropriate inhomogeneous terms are substituted into (4.12), we obtain the required condition on f, namely

$$\frac{\partial f}{\partial \tau} = -A\mu^2\nabla_H^2 f - B\nabla_H^4 f + C\nabla_H \cdot (|\nabla_H f|^2 \nabla_H f) + \xi D\tilde{\theta}|_1, \tag{4.13}$$

where $A = \frac{1}{45}$, $B = \frac{34}{231}$ and $C = \frac{10}{7}$. If the bounding planes are taken to be stress-free rather than rigid the equation obtained is the same as (4.13) but with different numerical values for the positive constants A, B, C (see CP).

The equation may be reduced to canonical form by writing

$$(\xi, \eta) = (A/B)^{\frac{1}{2}}(X, Y), \quad T = (A^2/B)\tau, \quad F = (C/B)^{\frac{1}{2}} f, \quad \tilde{F} = (C/B)^{\frac{1}{2}}\tilde{\theta}. \tag{4.14}$$

On substitution into (4.4), (4.13) we obtain

$$\left.\begin{aligned} \frac{\partial \tilde{F}}{\partial T} &= \gamma D^2 \tilde{F}, \\ \frac{\partial F}{\partial T} &= -\mu^2\nabla^2 F - \nabla^4 F + \nabla \cdot (|\nabla F|^2 \nabla F) + \gamma D\tilde{F}|_1 \end{aligned}\right\} \tag{4.15}$$

where $\gamma = B\zeta/A^2$, and $(\partial/\partial\xi, \partial/\partial\eta)$ is written as ∇ for simplicity. The boundary conditions on $\tilde{F}(\xi, \eta, z, T)$ are

$$\tilde{F} = 0, \quad z = 1+\lambda; \quad \tilde{F} = F_1, \quad z = 1. \tag{4.16}$$

Thus we have enormously simplified the original set of partial differential equations. The solution appears to depend on the three parameters μ, γ and λ. It will emerge, however, that stable *steady* states of the system depend on γ and λ only through $\beta \equiv \gamma\lambda^{-1}$; furthermore, if we consider a sequence of experiments in which one of μ or β is held fixed and the other varied, either parameter can be eliminated by a suitable co-ordinate transformation. In most of what follows we shall set $\mu = 1$, thus defining $\epsilon = (R - R_c)^{\frac{1}{2}}$.

5. Linearized theory

When the perturbations are of small amplitude, we may set the nonlinear term in (4.15) equal to zero. Then for $s > 0$ the solutions can be written

$$\left.\begin{aligned} F &= e^{sT} h(\xi, \eta), \\ \tilde{F} &= F \operatorname{sh}[(s/\gamma)^{\frac{1}{2}}(1+\lambda-z)]/\operatorname{sh}[(s/\gamma)^{\frac{1}{2}}\lambda] \end{aligned}\right\} \tag{5.1}$$

where $h(\xi, \eta)$ satisfies $\nabla^2 h = -\alpha^2 h$. Then

$$D\tilde{F}|_1 = -e^{sT}(s/\gamma)^{\frac{1}{2}} \coth[(s/\gamma)^{\frac{1}{2}}\lambda] \,.\, h \tag{5.2}$$

and so s satisfies the relation

$$s = \mu^2\alpha^2 - \alpha^4 - \beta G(\lambda(s/\gamma)^{\frac{1}{2}}), \tag{5.3}$$

where $\beta = \gamma\lambda^{-1}$ and where $G(x) = x \coth x$; analogous relations hold for $s < 0$. Since s is real, we may investigate the transition from stability to instability by setting $s = 0$. Then $G(x) = 1$ and the critical value of μ, μ_c, is given by

$$\mu_c^2 = \alpha^2 + \beta/\alpha^2 \tag{5.4}$$

so that the minimum Rayleigh number $R_{\min} = R_c + \epsilon^2\mu_{\min}^2$ is

$$R_{\min} = R_c + 2\epsilon^2\beta^{\frac{1}{2}} \tag{5.5}$$

and the critical wavenumber $\alpha = \alpha_c = \beta^{\frac{1}{4}}$.

These results agree with the asymptotic results obtained from the full linearized system. Thus when μ^2 is just greater than $2\beta^{\frac{1}{2}}$ the wavenumber of any growing mode is close to α_c. But if μ^2 is rather greater than this value then a whole band of modes are unstable, and the one that might be expected to appear in an experiment is that which has the maximum growth rate. Now in the classical problem $(\zeta \to \infty)$ these modes have wavenumbers similar to α_c. In the poorly conducting case, however, the position is quite different. We note that $s + \beta G(\lambda(s/\gamma)^{\frac{1}{2}})$ is a monotonically increasing function of s for $s > 0$. Thus the maximum growth rate $s_{\max}$ occurs when the terms in α in (5.3) are greatest, i.e. when

$$\alpha = \alpha_{\max} = \mu/\sqrt{2}. \tag{5.6}$$

Thus $\alpha_{\max}$ increases with μ and is independent of β. Clearly there is no relation between α_c and $\alpha_{\max}$ in general, and for small β, $\alpha_{\max}/\alpha_c$ can be very large. When the amplitude of the perturbation is finite, modes of different wavenumber interact and the realized steady states are only a very small subset of the modes permitted by the linearized theory. It was shown in CP that when $\zeta = 0$ the mode of maximum growth rate is not stable at finite amplitude, and that the stability of any mode is lost always to one of longer horizontal scale. Thus the dominant scale of convection increases with time until the largest scale available to the system is reached. When $\zeta \neq 0$ such an indefinite lengthening of scales cannot persist, since there is a small wavenumber cut-off. However, we can expect that, when $\beta \ll 1, \mu = O(1)$, the 'most stable mode' has wavelength much greater than the mode of maximum growth rate. Such questions of stability are virtually intractable for most nonlinear problems, and the difficulties are compounded in the case of convection in an infinite layer since solutions exist with many different patterns (rolls, square cells, hexagonal cells, etc.). A recent review (Busse 1978) describes the many ways in which degeneracy can be resolved by investigating the stability properties of the solutions; rigorous results have, however, been almost entirely confined to weakly nonlinear problems, where perturbation theory in powers of the amplitude can be utilized (Schlüter *et al.* 1965). In the present case the question of stability can be reduced to the discussion of the extrema of a certain functional, and it is this that we consider next.

6. The nonlinear variational principle

It has been suggested by S. Childress that equations of the type of (4.15) can be understood by considering the functional

$$V[F,\tilde{F}] = \left\langle \tfrac{1}{4}|\nabla F|^4 + \tfrac{1}{2}|\nabla^2 F|^2 - \tfrac{1}{2}|\nabla F|^2 + \frac{\gamma}{2}\int_1^{1+\lambda} (D\tilde{F})^2\, dz \right\rangle \tag{6.1}$$

where angle brackets denote an average over the fluid layer and we have set $\mu = 1$. It may quickly be established, by methods analogous to those of CP in the case $\gamma = 0$, that

$$\frac{dV}{dT} = -\left\langle \left(\frac{\partial F}{\partial T}\right)^2 + \int_1^{1+\lambda} \left(\frac{\partial \tilde{F}}{\partial T}\right)^2 dz \right\rangle \tag{6.2}$$

so that V decreases with time in any evolution of the system. Furthermore, the Euler–Lagrange equation for the stationary values of V with respect to F and $\tilde{F}$ at fixed time is precisely the time-independent version of (4.15). It is therefore clear that the stable steady solutions of (4.15) are those that correspond to local minima of V. For a steady solution, $\tilde{F} = F(1-\lambda^{-1}(z-1))$, so $D\tilde{F} = -F/\lambda$ and

$$\gamma\int_1^{1+\lambda} (D\tilde{F})^2\, dz = \beta F^2.$$

Thus the Newton's law of cooling model holds even for finite λ in the steady case. It can easily be shown that, when $\mu = 1$, the minimum of V is less than zero only when $\beta < \frac{1}{4}$, and this is the point at which the static solution $F \equiv 0$ loses stability. When F represents a stationary solution of (4.15), the following relation holds:

$$0 = \langle |\nabla F|^2 - |\nabla^2 F|^2 - |\nabla F|^4 - \beta F^2 \rangle. \tag{6.3}$$

Then substitution into (6.1) shows that in this case

$$V = -\tfrac{1}{4}\langle |\nabla F|^4 \rangle. \tag{6.4}$$

Furthermore, use of the Schwarz inequality in (6.1) reveals that

$$\left.\begin{aligned} V &\geqslant \tfrac{1}{4}\langle|\nabla F|^4\rangle - \tfrac{1}{2}\langle|\nabla F|^2\rangle \\ &\geqslant \tfrac{1}{4}\langle|\nabla F|^2\rangle^2 - \tfrac{1}{2}\langle|\nabla F|^2\rangle \\ &\geqslant -\tfrac{1}{4}. \end{aligned}\right\} \tag{6.5}$$

We shall see that this extreme lower bound is in fact approached as $\beta \rightarrow 0$. Since V is bounded below and continually decreases, the only asymptotic states at large times are steady ones.

There have been many attempts to understand the nature and heat-transport properties of convection in terms of a variational principle. The pioneering speculations of W. V. R. Malkus, and their interpretation by Howard (1963) centred on the idea that the convection would 'want' to maximize the heat transported across the layer or (which is equivalent) to maximize its thermal or viscous dissipation. Such a principle appears to have no general validity for convection (though, significantly, it gives a much more accurate estimate when the Prandtl number σ is effectively infinite). However, the present simplified system *does* obey such a principle: it maximizes $\langle|\nabla F|^4\rangle$! This quartic quantity is not directly related to the heat transport, although it can be related to quadratic quantities representing heat transport and viscous loss. Although it is not at all clear that there is a unique minimum of V, we seek in this study its absolute minimum, since the solution corresponding to this value is stable to all sufficiently small disturbances.

Before investigating the properties of V, we may prove quite simply that all two-dimensional steady solutions of (4.11) are unstable. Let F_0 be a steady solution (not necessarily two-dimensional) of (4.15); then F_0 gives a stationary value V_0 of V. Thus if F is perturbed slightly to $F_0 + \delta F$, (6.1) yields,

$$V = V_0 + \delta^2 V + O(|\delta F|^3),$$

where

$$\delta^2 V = \tfrac{1}{2}\langle 2(\nabla F_0 . \nabla(\delta F))^2 + |\nabla F_0|^2\,|\nabla(\delta F)|^2 + |\nabla^2(\delta F)|^2 + \beta|\delta F|^2 - |\nabla(\delta F)|^2\rangle. \tag{6.6}$$

Then the question of stability is decided by the sign of the minimum of $\delta^2 V/\langle|\delta F|^2\rangle$ as a functional of δF. If the minimum is non-negative, F_0 is stable, while if it is negative, F_0 is unstable.

Now suppose that $F_0 = F_0(\xi)$, representing a solution in the form of rolls. Then if we evaluate $\delta^2 V$ for $\delta F = CF_0(\eta)$ where C is a constant, we find that, since $\nabla F_0 . \nabla \delta F = 0$,

$$\delta^2 V = \tfrac{1}{2}C^2[\langle|F_0'|^2\rangle^2 + \langle F_0''^2\rangle + \beta\langle F_0^2\rangle - \langle F_0'^2\rangle], \tag{6.7}$$

where the primes denote derivatives with respect to the argument. But from (6.3)

$$\langle|F_0'|^4\rangle + \langle F_0''^2\rangle + \beta\langle F_0^2\rangle - \langle F_0'^2\rangle = 0 \tag{6.8}$$

and since $\langle F_0'^2\rangle^2$ is less than $\langle F_0'^4\rangle$ we can see that $\delta^2 V < 0$ for this value of δF. Thus the minimum of $\delta^2 V$ is also negative and roll solutions are unstable to three-dimensional perturbations. This result confirms and generalises the results of Busse & Riahi (1980) who found that when $\delta \equiv \frac{1}{4} - \beta \ll 1$ (so that the amplitude of convection is very small), *square* cells were the stable ones according to the theory of Schlüter *et al.* (1965). In the next section we shall see how their results emerge naturally from a consideration of V for small δ, and then extend the calculation numerically for $\delta = O(1)$.

7. Results

7.1. *Small-amplitude convection*

If we write $\delta = \frac{1}{4}-\beta$, $\mu = 1$ and $F = \delta^{\frac{1}{2}}G$ then (4.15) becomes, for steady solutions,

$$0 = \tfrac{1}{4}G+\nabla^4G+\nabla^2G-\delta G-\delta\nabla\,.\,(|\nabla G|^2\nabla G), \tag{7.1}$$

and we may seek a solution by expanding in powers of δ;

$$G = G_0+\delta G_1+\dots. \tag{7.2}$$

At leading order in δ,

$$0 = \tfrac{1}{4}G_0+\nabla^4G_0+\nabla^2G_0, \tag{7.3}$$

and this is solved by any function satisfying $\nabla^2G_0 = -\frac{1}{2}G_0$. At the next order in δ,

$$0 = \tfrac{1}{4}G_1+\nabla^4G_1+\nabla^2G_1-G_0-\nabla\,.\,(|\nabla G_0|^2\nabla G_0). \tag{7.4}$$

Now (7.3) is a self-adjoint system, and so a necessary and sufficient condition for a bounded solution of (7.4) is that the inhomogeneous term be orthogonal to *all* solutions of (7.3). Thus for any such solution, $\hat{G}$, say

$$\langle \hat{G}G_0\rangle = \langle \nabla\hat{G}\,.\,\nabla G_0|\nabla G_0|^2\rangle, \tag{7.5}$$

and in particular

$$\langle G_0^2\rangle = \langle|\nabla G_0|^4\rangle. \tag{7.6}$$

Thus if we substitute in a solution for G_0 (determined up to a constant by (7.3)), the amplitude can be found from (7.6) and thus the value of $V = -\frac{1}{4}\delta^2\langle|\nabla G_0|^4\rangle$ determined. Solutions with rectangular planform can be written in the form

$$G_0 = A\cos\alpha\xi\cos\beta\eta, \quad \text{where} \quad \alpha^2+\beta^2 = \tfrac{1}{2}; \tag{7.7}$$

then

$$\langle G_0^2\rangle = \tfrac{1}{4}A^2,$$

and

$$\langle|\nabla G_0|^4\rangle = \frac{A^4}{64}[\tfrac{5}{4}+4(\alpha^2-\beta^2)^2]. \tag{7.8}$$

Thus

$$V = -\tfrac{1}{16}\delta^2A^2 = -\delta^2[\tfrac{5}{4}+4(\alpha^2-\beta^2)^2]^{-1}, \tag{7.9}$$

so that the minimum of V occurs for square cells, yielding $V = -\frac{4}{5}\delta^2$. A similar calculation for roll solutions gives $V = -\frac{4}{9}\delta^2$, and it can be shown that hexagonal modes proportional to the eigenfunctions discovered by Christopherson (1940) also yield $V = -\frac{2}{3}\delta^2$. Thus for small δ square cells are indeed stable according to the 'V criterion', at least when δ is small.

7.2. *Numerical solutions for $\delta = O(1)$*

When β is not close to $\frac{1}{4}$ perturbation methods do not suffice, and the minimization problem for V must be solved numerically. The importance of the nonlinear work is that for finite-amplitude motions linearized theory permits a band of growing modes, and so a whole range of steady, finite-amplitude periodic solutions are permitted. Our minimization of V must therefore be with respect to the period of the solution as well as its functional form, and the result of the analysis is a value of the period for which the value of V is least, for a given planform. We then take this period as the 'stable' one, and suggest that it is the one most likely to occur when a series of experiments is conducted. No proofs are available, of course, that this 'most stable' period is the only stable one: experience from the perfectly conducting problem suggests that there are a range of stable periods. It does, however, give the first deductive method of finding the predominant wavelength.

To carry out the minimization, we represented F by its values F_i at the nodes of a Cartesian mesh, and evaluated the functional (6.1) using finite-difference methods. V was then minimized using a package from the NAG library. The periods in the X (and, for rectangular and hexagonal planforms, the Y) directions were also taken as independent variables. Figure 2 shows the domains and boundary conditions investigated. Figure 3 shows the results of the computation for rolls and square cells for various β between 0 and $\frac{1}{4}$. It can be seen that the value of V for square cells is less than for roll solutions, as already shown. V decreases as β decreases and, as expected, tends to $-\frac{1}{4}$ for $\beta \to 0$. An attempt was made to calculate V for hexagonal modes by the same method; but the boundary conditions used in the minimal rectangle, though permitting a hexagonal planform, also permitted other tesselations. In every case tried, the minimum found by the numerical routine was not of hexagonal type. Since the value of V was greater (for each β) than for either roll or square solutions, it is clear that hexagons are not a stable form of convection in this case. Figure 4 shows contour levels (for square cells) for $\beta = 0{\cdot}001$ for two values of α. There is a tendency for $|\nabla F|$ to approach unity when the amplitude is large, as predicted above.

In figure 5 we show the 'preferred wavelength' α_p of square-cell convection (curve (a)), plotted in terms of α, this value being *defined* as that which emerges from the minimization process as a function of β, compared with three other important wavelengths. These are (b) $\alpha_{\max}$, the mode of maximum growth rate, appearing as a horizontal line since it is independent of β; (c) the minimum α (maximum wavelength) at which convection can take place, calculated from (5.4) with $\mu = 1$; and (d) $\alpha_c = \beta^{\frac{1}{4}}$, the wavelength at which convection first occurs as R is increased. It will be seen that the preferred wavenumber α_p is very close to α_c except for small values of β. Numerical computations suggest that as $\beta \to 0$, $\alpha_p \sim \beta^{\frac{1}{3}}$; figure 6 shows $\alpha_p \beta^{-\frac{1}{3}}$ as a function of β. The $\frac{1}{3}$ exponent is supported by the asymptotic theory for $\beta \to 0$ given in appendix B. Thus for small β (or large μ, which as we have noted is completely equivalent) the preferred wavelength of convection is *larger* than stability theory might suggest.

It is difficult to relate this phenomenon of cell widening to any process of instability, since no direct time integration of the governing equations has yet been carried out. Krishnamurti (1970), studying rolls between *conducting* boundaries, found an increase in their width due primarily to 'dislocations' in the roll structure which tended to combine two rolls into one. At high Prandtl numbers and higher Rayleigh numbers 'spoke-pattern' convection occurs (see, for example, Busse 1978) and this leads to pronounced hysteresis in the observed wavenumber, with much lower wavenumbers being obtained if the Rayleigh number is decreased than vice versa. At low Prandtl numbers, by contrast, it is the skew-varicose instability that causes a decrease in wavenumber (Clever & Busse 1978). None of these routes seems applicable in the present problem. In CP it was shown that roll solutions can lose stability to larger rolls through a completely two-dimensional instability, so perhaps in the present case lengthening can occur via square cell modes. It is likely, however, that there is a range of values of L in a neighbourhood of L_p for which square cells of width L are stable to small disturbances.

We can see more clearly the way the wavelength changes with amplitude if we

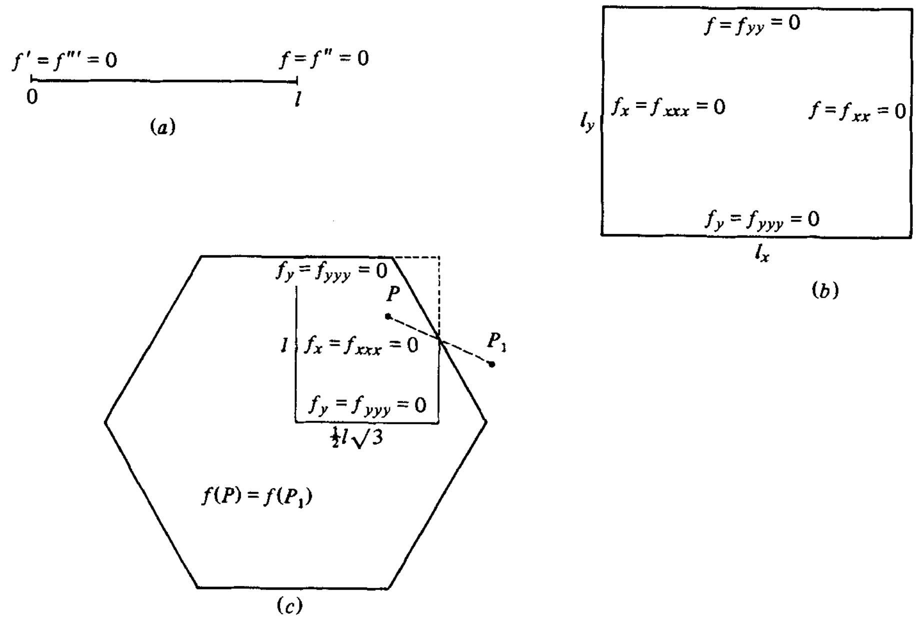

FIGURE 2. Domains of integration and boundary conditions appropriate to (*a*) rolls, (*b*) rectangular cells and (*c*) hexagonal cells.

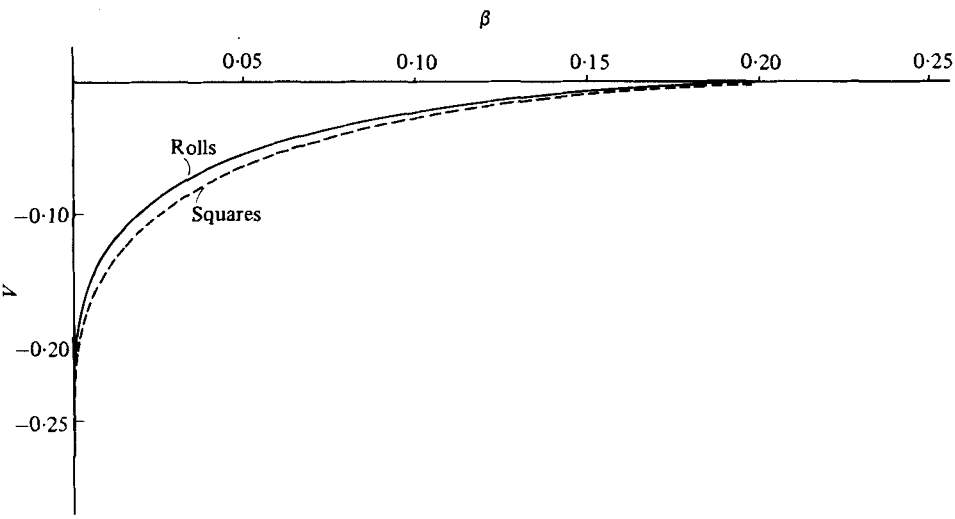

FIGURE 3. Graphs of the minimum of V as a function of β for roll and square planforms. There are no solutions for $\beta > \frac{1}{4}$.

write the results in terms of the physical variables R and ζ. Ignoring multiplication by constants of order unity, we have

$$\epsilon^2 = R - R_c, \quad L = 1/\epsilon\alpha, \quad \beta = \zeta\epsilon^{-4}. \tag{7.10}$$

Thus $L_c = 1/\epsilon\alpha_c = 1/\epsilon\beta^{\frac{1}{4}} = \zeta^{-\frac{1}{4}}$, independent of R (as is to be expected); while $L_p = 1/\epsilon\alpha_p = 1/\epsilon\beta^{\frac{1}{3}} = \zeta^{-\frac{1}{3}}\epsilon^{\frac{1}{3}} = \zeta^{-\frac{1}{3}}(R-R_c)^{\frac{1}{6}}$, at least when $(R-R_c) \geqslant \zeta^{\frac{1}{2}}$. Thus for fixed γ, L_p increases slowly with R as predicted. Although our problem and

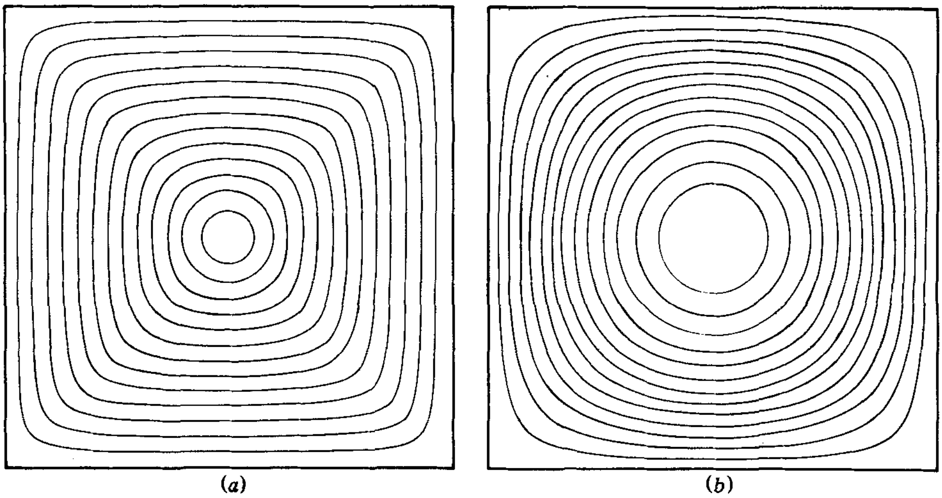

FIGURE 4. Contour levels for square cells with $\beta = 0{\cdot}001$ at (a) $\alpha = \alpha_p$, (b) $\alpha = \alpha_{min}$, the smallest wavenumber for convection to be possible.

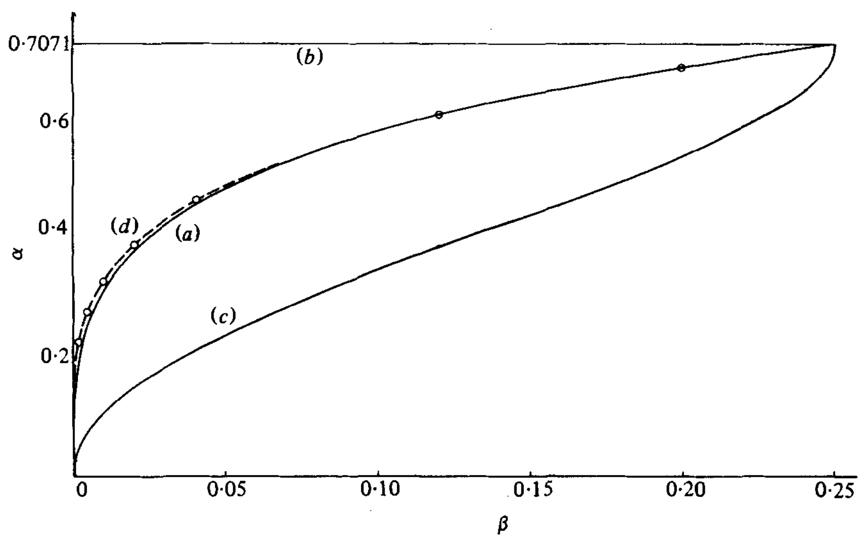

FIGURE 5. (a) The preferred wavenumber α_p compared with (b) α_{max}, the scale of maximum linear growth rate, (c) α_{min}, the smallest wavenumber for convection, and (d) $\alpha_c = \beta^{\frac{1}{4}}$, the 'most unstable' wavenumber.

Krishnamurti's are not directly comparable, the general picture of the dependence of L_p on R is very similar to Krishnamurti's figure 9.

Finally, it must be emphasized that all minimization techniques are local in nature, and that there is never a guarantee of a unique solution. We believe that we have isolated the minima corresponding to roll and rectangular (i.e. in this case square) planforms. It is quite possible that from general initial conditions non-regular tesselations could result, and this is particularly likely if side walls are near enough to influence the pattern significantly. The effect of distant side walls can be discussed using a WKB-like method as discussed by Segel (1969), but complicated planforms of the type observed by experimenters (see for example Busse 1978) probably have

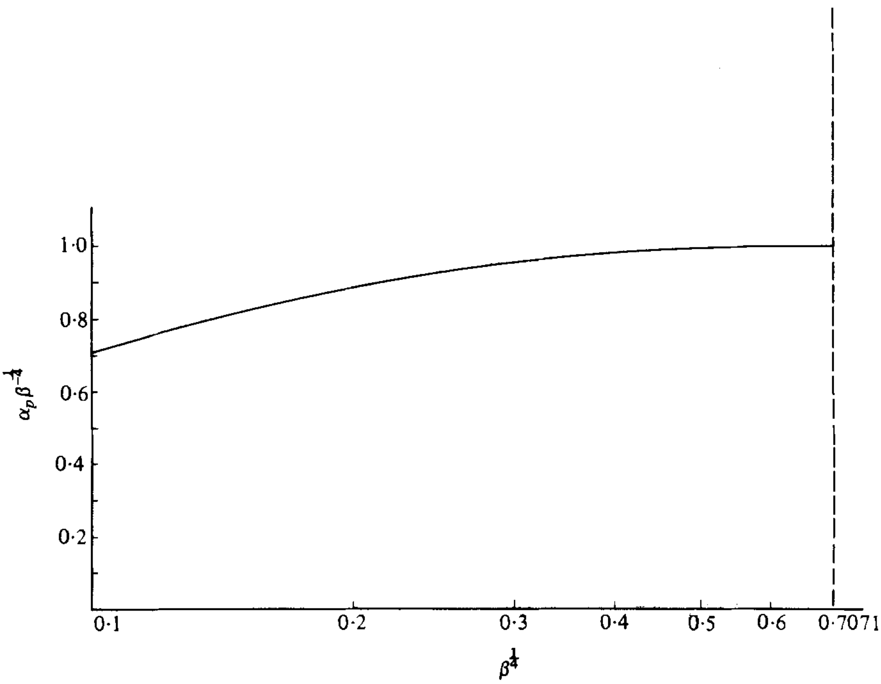

FIGURE 6. Plot of $\alpha_p\beta^{-\frac{1}{4}} \equiv \alpha_p/\alpha_c$ as a function of $\beta^{\frac{1}{4}}$.

a local domain of attraction. Current computing power allows a numerical simulation of a 'box' containing many wavelengths, and an investigation of this type is currently in progress.

8. Conclusion

In previous sections we have shown that thermal convection between poorly conducting boundaries can be investigated by 'shallow water theory' methods of the type pioneered for convection problems by Childress & Spiegel (1981). The method yields an evolution equation for the temperature perturbation in the two horizontal co-ordinates and time, and this equation in turn can be understood in terms of a variational principle. It can then be shown that cells of square planform are the most stable among regular tesselated patterns, even when the amplitude of the temperature perturbation is $O(1)$ so that the convection is fully nonlinear. This result confirms and extends the results of Busse & Riahi (1980) who determined the stability properties very close to the critical level of the Rayleigh number for the onset of convection. The calculations yield for the first time a prediction based on analysis that the preferred wavelength of convection increases slowly with amplitude.

There has been no mention thus far of the possible effect of finite Prandtl number σ on the results. Although the analysis is formally valid only for infinite σ, it can be shown, by methods analogous to that of CP, that even when $\sigma = O(1)$ it never enters the calculations to the order that we have taken them. The effect is due to the symmetry between the top and bottom boundaries. As noted in CP, if the velocity boundary conditions are different at $z = \pm 1$, an extra term appears in the evolution equation that *does* depend on σ, but since this new equation cannot be discussed simply in terms of a variational principle we do not consider it here.

When $R-R_c$ is of order unity, the analysis ceases to be accurate, and one might then expect the dominant horizontal scales to be of the order of the layer depth. However, recent computations of the full nonlinear system with $\zeta = 0$ and no y-dependence (Hewitt, McKenzie & Weiss 1980) show that long wavelength motions can persist well into the nonlinear regime, even when the thermal variation takes place in thin boundary layers. In that paper and in Chapman *et al.* (1980) it is noted that calculations for small Biot number have an important bearing on the study of convection in the Earth's upper mantle. New information on the planform of mantle convection is also very suggestive of a 'square cell' structure (McKenzie *et al.* 1980). Of course, the Biot number in the earth is not very small: and one of the most entertaining and significant problems raised by the present work is that of determining the value of γ at which the change-over between roll and square-cell solutions takes place.

I am indebted to S. Childress who suggested the 'shallow water theory' method, to D. O. Gough and M. J. D. Powell for advice on numerical methods, and to C. J. Chapman for many illuminating discussions.

Appendix A. Solution of the linear eigenvalue problem (3.1)

It may quickly be established that the solution of (3.1a), namely

$$\hat{\theta}(z) = \hat{\theta}(1)\frac{\sinh\alpha(1+\lambda-z)}{\sinh\lambda\alpha} \tag{A 1}$$

and similarly for the lower slab, leads to the boundary condition on $\hat{\theta}$:

$$D\hat{\theta}|_{\pm1} = \mp\,\zeta\alpha\coth\alpha\,\hat{\theta}|_{\pm1}. \tag{A 2}$$

The eigenvalue $R_0(\alpha,\zeta)$ can then be calculated as the minimum of the homogeneous functional

$$R_0 = \frac{\langle\hat{\theta}[(\alpha^2-D^2)\hat{\theta}]\rangle\langle[(D^2-\alpha^2)\hat{\phi}]^2\rangle}{\alpha^2\langle\hat{\theta}\hat{\phi}\rangle^2}. \tag{A 3}$$

R_{min} was found by representing $\hat{\theta}$ and $\hat{\phi}$ as polynomials in z^2, constraining some of the coefficients to satisfy (A 2) and the boundary conditions (2.11) on $\hat{\phi}$, and finding the minimum as a function of the remaining coefficients and α, using a standard minimization routine from the NAG subroutine library.

Appendix B. The square-cell tesselation for small β, and the minimum value of V

We consider steady solutions of equation (4.15) with $\mu = 1$, which can be written

$$0 = \nabla^4F+\nabla^2F+\beta F-\nabla\,.\,(|\nabla F|^2\nabla F), \tag{B 1}$$

in a square cell of side l, for $\beta \ll 1$. If we write $F = F_0+\beta F_1$, substitute into (B 1) and expand in powers of β, we obtain at leading order

$$0 = \nabla^4 F_0 + \nabla^2 F_0 - \nabla . (|\nabla F_0|^2 \nabla F_0), \tag{B 2a}$$

$$0 = \nabla^4 F_1 + \nabla^2 F_1 - 2\nabla . ((\nabla F_0 . \nabla F_1) . \nabla F_0) - \nabla . (|\nabla F_0|^2 \nabla F_1) + F_0, \tag{B 2b}$$

so that F_0 is the solution of the 'perfect' problem. For steady solutions the functional V can be written

$$V = -\tfrac{1}{4}\langle|\nabla F|^4\rangle = -\tfrac{1}{4}\langle|\nabla F_0|^4\rangle - \beta\langle|\nabla F_0|^2 \nabla F_0 . \nabla F_1\rangle + \dots, \tag{B 3}$$

and if (B 2a) is multiplied by F_1, (B 2b) by F_0, the resulting equations averaged and combined to eliminate terms bilinear in F_0 and F_1, we can rewrite (B 3) as

$$V = -\tfrac{1}{4}\langle|\nabla F_0|^4\rangle + \tfrac{1}{2}\beta\langle F_0^2\rangle + \dots. \tag{B 4}$$

We may thus consider only (B 2a) from now on. Suppose that the origin $(\xi, \eta) = (0, 0)$ is at the centre of the cell, and that $F = 0$ on the boundaries, $\xi = \pm\tfrac{1}{2}l; \eta = \pm\tfrac{1}{2}l$. Then a solution that is continuous everywhere, satisfies the equation everywhere except at $\xi = \pm\eta$, and satisfies all the boundary conditions is

$$F_0 = \tfrac{1}{2}l - \tfrac{1}{2}|\xi - \eta| - \tfrac{1}{2}|\xi + \eta|. \tag{B 5}$$

Examination of the numerical solutions reveals that F_0 differs from this form only in transition regions of thickness $O(1)$ near $\xi = \pm\eta$. $|\nabla F|$ is of order unity throughout, and is almost equal to 1 almost everywhere away from these regions. Thus we might expect, and computations confirm, that for large l

$$\langle|\nabla F_0|^4\rangle \sim 1 - A/l + O(l^{-2}) \tag{B 6}$$

and the constant A is about 4. In calculating the second term in (B 4) the solution (B 5) may be used at leading order: the integration is elementary, and leads to the approximate expression

$$V = -\frac{1}{4} + \frac{1}{l} + \frac{\beta l^2}{48}. \tag{B 7}$$

Thus the minimum value of V is about $-\tfrac{1}{4} + 0{\cdot}52\beta^{\frac{1}{3}}$ ($\beta \ll 1$) and occurs for $l \simeq 2{\cdot}9\beta^{-\frac{1}{3}}$. These values agree well with the computations.

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